Maximal Ideals of Triangular Operator Algebras

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Let $\mathcal{H} := \ell^2(\mathbb{N})$ and let $\{e_k\}_{k=1}^{\infty}$ be the standard basis. Let $\mathcal{T}$ be the algebra of all (bounded) operators which are upper triangular with respect to $\{e_k\}$.

Question

What are the maximal two-sided ideals of $\mathcal{T}$?

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Observe that $\mathcal{D}$, the set of diagonal operators w.r.t. $\{e_k\}$ is *-isomorphic to $\ell^\infty(\mathbb{N})$, so we identify them. Write $\mathcal{S}$ for the set of *strictly* upper triangular operators w.r.t. $\{e_k\}$.

Fact

Let $\mathcal{M}$ be a maximal ideal of $\ell^\infty(\mathbb{N})$ and let $\mathcal{J} := \mathcal{M} + \mathcal{S}$. Then $\mathcal{J}$ is a maximal ideal of $\mathcal{T}$.

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Write $\Delta(T)$ for the diagonal part of T. Suppose $T \notin \mathcal{J}$.

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Write $\Delta(T)$ for the diagonal part of T. Suppose $T \notin \mathcal{J}$.
$T - \Delta(T) = J \in \mathcal{S} \subseteq \mathcal{J}$ and so $\Delta(T) \notin \mathcal{J}$, hence $\Delta(T) \notin \mathcal{M}$. Thus $D\Delta(T) + M = I$ and so $D(T - J) + M = I \in \langle T, \mathcal{J} \rangle$. □

The maximal ideals of $\ell^\infty(\mathbb{N})$ are points in $\beta\mathbb{N}$, the Stone-Cech compactification of $\mathbb{N}$, so this would give a good description of the maximal ideals of $\mathcal{T}$.

Question

Are all the maximal ideals of $\mathcal{T}$ of the form $\mathcal{M} + \mathcal{S}$ where $\mathcal{M}$ is a maximal ideal of $\ell^\infty(\mathbb{N})$?

Proposition

TFAE:

1. *All the maximal ideals of* $\mathcal{T}$ *are of the form* $\mathcal{M} + \mathcal{S}$.
2. *All the maximal ideals of* $\mathcal{T}$ *contain* $\mathcal{S}$.
3. *No proper ideal of* $\mathcal{T}$ *contains an operator* $I + S$, $(S \in \mathcal{S})$.

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(1) $\Rightarrow$ **(2)** $\Rightarrow$ **(3):** Obvious.

(3) $\Rightarrow$ **(2):** Contrapositive. Suppose $\mathcal{J} \not\supseteq \mathcal{S}$ is a maximal ideal of $\mathcal{T}$. Then $\mathcal{J} + \mathcal{S} = \mathcal{T}$ and so $I = J - S$. □

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(2) $\Rightarrow$ (1): Let $\mathcal{J}$ be a maximal ideal of $\mathcal{T}$. Since $\mathcal{J} \supseteq \mathcal{S}$, then also $\mathcal{J} \supseteq \Delta(\mathcal{J})$. But $\Delta(\mathcal{J}) \lhd \mathcal{D}$ so let $\mathcal{M} \supseteq \Delta(\mathcal{J})$ be a maximal ideal of $\mathcal{D}$ and we saw $\mathcal{M} + \mathcal{S}$ is a maximal ideal of $\mathcal{T}$ – that contains $\mathcal{J}$.

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Just to be clear, an operator X fails to belong to a proper ideal of $\mathcal{T}$ iff we can find $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ such that

$$A_1 X B_1 + \cdots + A_n X B_n = I$$

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Not so in infinite dimensions.

Let $\begin{bmatrix} 0 & 1 & 0 & & & \\ & 0 & 1 & 0 & & \\ & & 0 & 1 & 0 & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$ be the unilateral backward shift

Then $I - U = \begin{bmatrix} 1 & -1 & 0 & & & & \\ 0 & 1 & -1 & 0 & & & \\ & 0 & 1 & -1 & 0 & & \\ & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$ is not invertible

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Let $\sigma \subseteq \mathbb{N}$ and let

$$P_\sigma := \operatorname{Proj}\left(\overline{\operatorname{span}}\{e_k \ : \ k \in \sigma\}\right)$$

Note $UP_{2\mathbb{N}} = P_{2\mathbb{N}-1}U$ and $UP_{2\mathbb{N}-1} = P_{2\mathbb{N}}U$

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This simple observation connects us to a famous open problem known as The Kadison-Singer problem or The Paving Problem.

Let the standard atomic masa, $\mathcal{D}$, and the projections, P_σ, be as defined before.

Definition

Say that $X \in B(\mathcal{H})$ can be "paved" if, given any $\epsilon > 0$, there are pwd sets $\sigma_1, \dots \sigma_n \subseteq \mathbb{N}$ such that

$$\sigma_1 \cup \cdots \cup \sigma_n = \mathbb{N}$$

and

$$\left\| \Delta(X) - \sum_{k=1}^{n} P_{\sigma_k} X P_{\sigma_k} \right\| < \epsilon$$

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Question (Paving Problem)

Can every operator in $B(\mathcal{H})$ be paved?

Proposition

If every operator can be paved, then no operator of the form $I + S$ $(S \in \mathcal{S})$ can belong to a proper ideal of $\mathcal{T}$.

Proof.

$I + S$ can be paved by projections in $\mathcal{D}$. So

$$\left\| I - \sum_{k=1}^{n} P_{\sigma_i}(I+S)P_{\sigma_i} \right\| < 1$$

and $\sum_{k=1}^{n} P_{\sigma_i}(I+S)P_{\sigma_i}$ is invertible in $\mathcal{T}$. □

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Question (Kadison-Singer)

Let $\mathcal{D}$ be an atomic masa in $B(\mathcal{H})$. Does every pure state of $\mathcal{D}$ have a unique extension to a state of $B(\mathcal{H})$?

- If $\mathcal{M}$ is a non-atomic masa in $B(\mathcal{H})$ (i.e. $L^\infty(0,1)$) then it has pure states with non-unique extensions [KS59]. (In fact *no* pure states on $L^\infty(0,1)$ extend uniquely [And79a].)
- If $\mathcal{D}$ is an atomic masa in $B(\mathcal{H})$ (i.e. $\ell^\infty(\mathbb{N})$) and ϕ is a pure state on $\mathcal{D}$, then $\phi \cdot \Delta$ is a state on $B(\mathcal{H})$. (Anderson [And79b] showed it is a *pure* state.)
- Is $\phi \cdot \Delta$ the *only* extension of ϕ to a state of $B(\mathcal{H})$?

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Proof.

(1) $\Rightarrow$ (2): Let $\hat{\phi}$ be a state extension of ϕ. Then $\hat{\phi}$ is a $\mathcal{D}$-bimodule map. Thus by paving X we can arrange

$$\phi \cdot \Delta(X) = \hat{\phi} \cdot \Delta(X) \sim_\epsilon \hat{\phi}\left(\sum_{k=1}^{n} P_{\sigma_i} X P_{\sigma_i}\right) = \sum_{k=1}^{n} \phi(P_{\sigma_i})^2 \hat{\phi}(X) = \hat{\phi}(X)$$

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Lemma

$\hat{\phi}$ is a $\mathcal{D}$-bimodule map.

Proof.

Let $p \in \mathcal{D}$ be a projection. Then $\hat{\phi}(p) = \phi(p) = \phi(p)^2 = 0, 1$. If $\phi(p) = 0$ then by Cauchy-Schwartz,

$$\hat{\phi}(px) = 0 = \hat{\phi}(p)\hat{\phi}(x)$$

If $\phi(p) = 1$ then, again by Cauchy-Schwartz,

$$\hat{\phi}(px) = \hat{\phi}(x) - \hat{\phi}(p^{\perp}x) = \hat{\phi}(x) = \hat{\phi}(p)\hat{\phi}(x)$$

(Extend to arbitrary $a \in \mathcal{D}$ by spectral theory.) □

- Reid; [Rei71]
- Anderson; [And79a, And79b]
- Berman, Halpern, Kaftal, Weiss; [BHKW88]
- Bourgain, Tzafriri; [BT91]
- Weaver; [Wea04, Wea03]
- Casazza, Christensen, Lindner, Vershynin; [CCLV05]
- Casazza, Tremain "The paving conjecture is equivalent to the paving conjecture for triangular matrices"; [CT]

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Proof.

If $AXB = I$ let $P_n := P_{\{1,\dots,n\}}$ and note

$$P_n = (P_nAP_n)(P_nXP_n)(P_nBP_n) = (P_nBAP_n)P_nXP_n$$

since P_nBP_n is the (two-sided) inverse of P_nAXP_n in $P_n\mathcal{H}$. Taking WOT-limits we see $BAX = I$ and similarly $XBA = I$. □

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Proposition

Let $X \in \mathcal{T}$. There are $A, B \in \mathcal{T}$ with $AXB = I$ iff X is an invertible operator.

So how about solving $AXB + CXD = I$?

First express as a finite dimensional problem:

Question

Given an $n \times n$ matrix $X = I + S$ (S strictly upper triangular), can we find upper triangular matrices $A, \ldots, D$ such that

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where the $\max\{\|A\|, \ldots, \|D\|\}$ is bounded in terms of $\|X\|$ but independently of n?

Lemma

Let $X = I + S \in M_n(\mathbb{C})$ *where* S *is strictly upper triangular. Then there are* $A, \ldots, D \in M_n(\mathbb{C})$ *such that* $AXB + CXD = I$ *and* $\max\{\|A\|, \ldots, \|D\|\} \le \|X\|$.

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$$1 = \det X < \|X\|^{n/2} / \|X\|^{n/2} \leq 1.$$

Thus the first $n/2$ of the s_i are at least $\|X\|^{-1}$.

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Thus the first $n/2$ of the s_i are at least $\|X\|^{-1}$. There are o.n. bases $u_i, v_i (1 \leq i \leq n)$ such that $Xu_i = s_i v_i$. Let A, B be matrices mapping $v_i \mapsto (1/s_i)e_i$ and $e_i \mapsto u_i$ for $1 \leq i \leq n/2$. Then AXB is the projection onto $\operatorname{span}\{e_1, \ldots e_{\frac{n}{2}}\}$ and $\|A\|, \|B\| \leq s_{\frac{n}{2}}^{-1} \leq \|X\|$.

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At least we see there is no spectral obstruction to a two-term decomposition. Might there be other obstructions? Index perhaps?

Question

Given $X = I + S$ $(S \in \mathcal{S})$, are there $A, \ldots, D \in \mathcal{T}$ such that $AXB + CXD = I$?

Suppose now that there *is* a maximal ideal $\mathcal{J}$ of $\mathcal{T}$ that contains $X = I + S$ $(S \in \mathcal{S})$ and deduce some consequences.
Let

$$\Sigma = \{\sigma \subseteq \mathbb{N} \ : \ I - P_\sigma \in \mathcal{J}\}$$

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Then

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2. Σ *contains all cofinite subset of* $\mathbb{N}$.
3. $\sigma \in \Sigma \ \Rightarrow \ \sigma + 1 \in \Sigma$.
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Proof.

If $\sigma \in \Sigma$ and $\tau \supseteq \sigma$ then $P_{\tau^c} = P_{\tau^c} P_{\sigma^c} \in \mathcal{J}$.
If $\sigma_1, \sigma_2 \in \Sigma$ then $P^{\perp}_{\sigma_1 \cap \sigma_2} = P_{\sigma_1^c \cup \sigma_2^c} = P_{\sigma_1^c} + P_{\sigma_2^c} - P_{\sigma_1^c} P_{\sigma_2^c}$. □

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Proof.

For each k, $P_{\{k\}} = P_{\{k\}} X P_{\{k\}} \in \mathcal{J}$ so $\{k\}^c \in \Sigma$, a filter. □

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Proof.

$\mathcal{J} \not\supseteq \mathcal{S}$ and so $\mathcal{S} + \mathcal{J} = \mathcal{T}$. Let U be the backward shift. Then $U\mathcal{T} = \mathcal{T}U = \mathcal{S}$ and so U is invertible $(\text{mod})\mathcal{J}$. But $UP_{\sigma+1} = P_\sigma U$ so $P_\sigma = I(\text{mod})\mathcal{J}$ iff $P_{\sigma+1} = I(\text{mod})\mathcal{J}$. □

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Proof.

Neither $2\mathbb{N}$ nor $2\mathbb{N} - 1$ can be in Σ for then its complement is in Σ also. □

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Nest algebras

Definition (Ringrose, [Rin65])

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{N}$ a complete chain of subspaces containing 0 and H. This is called a nest. Define the nest algebra, $\mathrm{Alg}(\mathcal{N})$, for a given nest $\mathcal{N}$ to be

$$\mathrm{Alg}(\mathcal{N}) := \{X \in B(\mathcal{H}) \ : \ XN \subseteq N \quad \forall N \in \mathcal{N}\}$$

See Davidson, *Nest Algebras*, [Dav88].

Nest algebras

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See Davidson, *Nest Algebras*, [Dav88].

Example

Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{C}^n$. Let $N_i := \text{span}\{e_1, \ldots, e_i\}$ and $\mathcal{N} := \{0, N_i \ : \ 1 \leq i \leq n\}$.
Then $\text{Alg}(\mathcal{N}) = T_n(\mathbb{C})$.

Example

Let e_i $(i \in \mathbb{N})$ be the standard basis for $\mathcal{H} = \ell^2(\mathbb{N})$. Let $N_i := \operatorname{span}\{e_1, \ldots, e_i\}$ and $\mathcal{N} := \{0, N_i, \mathcal{H} \ : \ i \in \mathbb{N}\}$.
Then $\operatorname{Alg}(\mathcal{N})$ is the algebra of all bounded operators which are upper triangular w.r.t. $\{e_i\}$.
In other words,

$$\operatorname{Alg}(\mathcal{N}) = \mathcal{T}$$

The Volterra Nest

Example

Let $H = L^2(0,1)$. For each $t \in [0,1]$ let

$$N_t := \{f \in L^2(0,1) \ : \ f \text{ is supported a.e. on [0,t]}\}$$

In other words, $P(N_t)$ is multiplication by $\chi_{[0,t]}$. Clearly $\mathcal{N} := \{N_t \ : \ t \in [0,1]\}$ is a nest.

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Remark

$\mathrm{Alg}(\mathcal{N})$ contains the Volterra integral operator,

$$f \longmapsto \int_x^1 f(t)\,dt$$

Classification of nest algebras

Theorem (Ringrose, [Rin66])

Let $\phi : \mathrm{Alg}(\mathcal{N}_1) \to \mathrm{Alg}(\mathcal{N}_2)$ *be an algebraic isomorphsim. Then there is an invertible operator* $S \in B(\mathcal{H}_1, \mathcal{H}_2)$ *such that*

$$\phi(T) = STS^{-1} = \mathrm{Ad}_S(T) \text{ for all } T \in \mathrm{Alg}(\mathcal{N}_1)$$

Now $\phi = \mathrm{Ad}_S$ iff $\{SN : N \in \mathcal{N}_1\} = \mathcal{N}_2$. So classifying nest algebras up to isomorphism means classifying nests up to similarity.

Theorem (Erdos, [Erd67])

Nests are completely classified up to unitary equivalence by

- *An order type*
- *A measure class, and*
- *A multiplicity function*

C.f. Unitary invariants for bounded selfadjoint operators (spectrum, measure class, mutliplicity function).

Question

Any similarity transform preserves order type. Must it also preserve multiplicity and/or measure class?

Let $\mathcal{N}$ be the Volterra nest on $\mathcal{H} = L^2(0,1)$. I.e. $\mathcal{N} = \{N_t \ : \ t \in [0,1]\}$ where

$$N_t = \{f \ : \ f(x) = 0 \text{ a.e. } x \notin [0,t]\}$$

Example

The map $N_t \longmapsto N_t \oplus N_t$ preserves order type and measure class, but not spectral multiplicity.

Example

Let $f : [0,1] \to [0,1]$ be increasing, bjijective, *not* absolutely continuous. The map $N_t \longmapsto N_{f(t)}$ preserves order type and multiplicity, but not measure class.

Theorem (Davidson, [Dav84])

Let $\mathcal{N}_1, \mathcal{N}_2$ be nests and $\theta : \mathcal{N}_1 \to \mathcal{N}_2$ be and order isomorphism. There is an invertible operator S such that

$$\theta(N) = SN \quad \textit{for all } N \in \mathcal{N}_1$$

iff θ is dimension-preserving, i.e. if

$$\dim \theta(N) \ominus \theta(M) = \dim N \ominus M \ \textit{ for all } M < N \textit{ in } \mathcal{N}_1$$

Corollary

Both of the previous two examples are implemented by invertibles!

Corollary

Nest algebras are classified up to isomorphism by "order-dimension" type.

- Proof uses Voiculescu's notion of approximate unitary equivalence.
- Based on N. T. Andersen's study of unitary equivalence of quasi-triangular algebras
- Slightly earlier result of D. Larson [Lar85] showed all continuous nests are similar.

Proposition

The commutator ideal of a continuous nest is the whole algebra.

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Proof.

By the Similarity Theorem, $\mathrm{Alg}(\mathcal{N}) \cong \mathrm{Alg}(\mathcal{N} \oplus \mathcal{N}) = M_2(\mathrm{Alg}(\mathcal{N}))$ and so

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]^2 = I$$

□

Corollary

Let $\mathcal{N}$ be the Volterra nest. Then there is no ideal $\mathcal{S} \lhd \mathrm{Alg}(\mathcal{N})$ such that $\mathrm{Alg}(\mathcal{N}) = \mathcal{D}(\mathcal{N}) \oplus \mathcal{S}$.

Corollary

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Proof.

$\mathcal{D}(\mathcal{N}) = \mathcal{N}' = \mathcal{N}''$ is abelian so $\mathcal{S}$ would contain the commutator ideal. □

Proposition

$\mathrm{Alg}(\mathcal{N})$ *has non-zero idempotents which are "zero on the diagonal", i.e.*

$$P(N_{b_i} - N_{a_i})\, Q\, P(N_{b_i} - N_{a_i}) = 0 \text{ where } \sum_i P(N_{b_i} - N_{a_i}) = I$$

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Proof.

Write the Cantor middle-$\frac{1}{3}$ set as $K = [0,1] \setminus \bigcup_{i=1}^{\infty}(a_i, b_i)$. Let $f : [0,1] \to [0,1]$ map K to a non-null set. By the Similarity Theorem, $SN_t = N_{f(t)}$. Let $P = M_{\chi_{f(K)}}$ and $Q = SPS^{-1}$. □

Interpolation Theorem

Let $\mathcal{N}$ be the Volterra nest. For a Borel set $S \subseteq [0,1]$ write $E(S) = M_{\chi_S}$. Define the diagonal seminorm

$$i_x(T) := \inf\{\|P(N_x \ominus N_t)TP(N_x \ominus N_t)\| \ : \ t < x\}$$

Theorem (Interpolation Theorem, [Orr95])

Let $T \in \mathrm{Alg}(\mathcal{N})$, $a > 0$, *and*

$$S := \{x \ : \ i_x(T) \geq a\}$$

Then there are $A, B \in \mathrm{Alg}(\mathcal{N})$ *such that* $ATB = E(S)$.

Proof uses:

- Larson-Pitts [LP91] classification of idempotent equivalence
- Construction of "zero-diagonal" idempotents which sum to an idempotent that is equivalent to $E(S)$
- Factorization of "zero-diagonal" idempotents through T

Corollary

Let $\mathcal{N}$ be a continuous nest and $X \in \mathrm{Alg}(\mathcal{N})$. TFAE:

1. *There are $A_1 \dots, A_n$ and $B_1, \dots, B_n$ in $\mathrm{Alg}(\mathcal{N})$ such that*

$$A_1 X B_1 + \cdots + A_N X B_n = I.$$

   I.e. X does not belong to any proper ideal of $\mathrm{Alg}(\mathcal{N})$.

2. *There are $A, B \in \mathrm{Alg}(\mathcal{N})$ such that $AXB = I$.*
3. *$i_t(X) \geq a > 0$ for all $0 \leq t \leq 1$.*
   I.e.

$$\inf\{\|P(N_t \ominus N_s) T P(N_t \ominus N_s)\| \;:\; 0 \leq s < t \leq I\} > 0$$

Corollary

Let $\mathcal{N}$ be a continuous nest and $X \in \text{Alg}(\mathcal{N})$. TFAE:

1. *There are $A_1 \ldots, A_n$ and $B_1, \ldots, B_n$ in $\text{Alg}(\mathcal{N})$ such that*

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   I.e.

   $$\inf\{\|P(N_t \ominus N_s) T P(N_t \ominus N_s)\| \ : \ 0 \leq s < t \leq I\} > 0$$

Compare this with $\mathcal{T}$ where:

- 3. is analgous to $X = I + S$
- We saw 1. $\not\Leftrightarrow$ 2.
- We could not settle whether a version of 2. with *two* terms might be possible.

Consequences of the Interpolation Theorem include:

- Identification of maximal off-diagonal ideals and constructions of maximal triangular algebras [Orr95]
- Classification of the maximal ideals of continuous nest algebra and the lattice they generate [Orr94]
- The invertibles are connected in many nest algebras [DO95, DOP95]
- Description of epimorphisms of nest algebras [DHO95]
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Question

Does there exist an epimorphism $\phi : \mathcal{T} \to B(\mathcal{H})$?

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Fact

If so, then $\ker \phi$ *contains an operator* $I + S$ $(S \in \mathcal{S})$.

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Does there exist an epimorphism $\phi : \mathcal{T} \to B(\mathcal{H})$?

Fact

If so, then $\ker\phi$ *contains an operator* $I + S$ $(S \in \mathcal{S})$*.*

Proof.

The commutator ideal of $\mathcal{T}$ is $\mathcal{S}$ and the commutator ideal of $B(\mathcal{H})$ is $B(\mathcal{H})$. Thus $\phi(S) = I = \phi(I)$ and so $I - S \in \ker\phi$. □

Definition

The Bass stable rank of an algebra is the smallest n such that whenever $(g_1, \ldots, g_{n+1})$ generate the algebra as a left-ideal then we can find a_i such that

$$(g_1 + b_1 g_{n+1}, g_2 + b_2 g_{n+1}, \ldots g_n + b_n g_{n+1})$$

also generate the algebra as a left ideal.

Question

What is the Bass stable rank of $\mathcal{T}$?

Theorem (Arveson, [Arv75])

$G_1, \ldots, G_n$ generate $\mathcal{T}$ as a left ideal iff

$$G_1^* P_k^\perp G_1 + \cdots + G_n^* P_k^\perp G_n \geq a P_k$$

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