Classifying Stable Ideals of Nest Algebras

John L. Orr

jorr@math.unl.edu

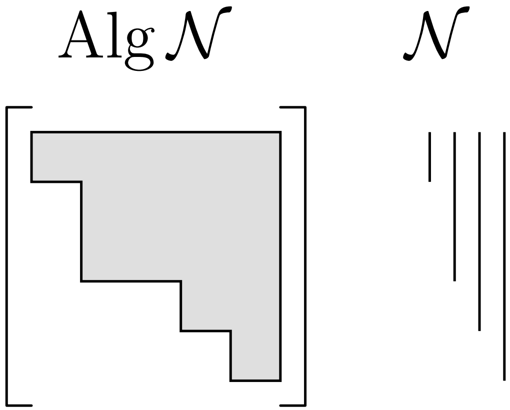
University of Nebraska–Lincoln

Introduction

Lecture plan:

- Nest algebras and their ideals
- Stable ideals
- Examples
- Characterization
- Classification
- Applications

Nest Algebras



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$$\operatorname{Alg}\mathcal{N} = \{X : N^{\perp} X N = 0\}$$

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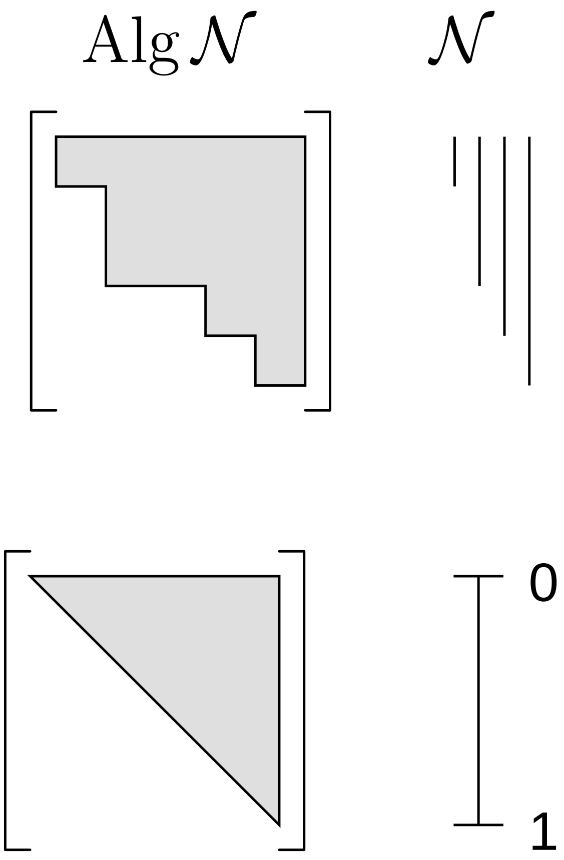

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Mostly, we use *continuous* nests.

Ideals

There is a very rich selection of norm-closed ideals.

- Weakly closed ideals
- Radicals
- Compact and compact-like

Weakly Closed Ideals

Theorem 1 (Erdos-Power, '82). *$\mathfrak{J}$ is a weakly closed ideal of $\operatorname{Alg}\mathcal{N}$ if and only if there is a increasing map $\theta : \mathcal{N} \to \mathcal{N}$ satisfying $\theta(N) \le N$ such that*

$$\mathfrak{J} = \{X \in \operatorname{Alg}\mathcal{N} \ : \ \theta(N)^{\perp} X N = 0 \quad \forall N \in \mathcal{N}\}$$

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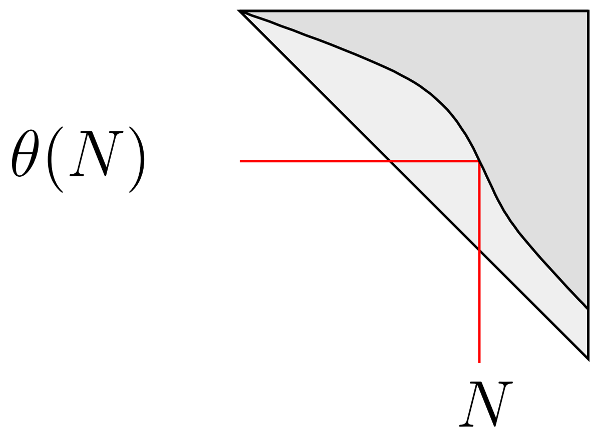

The Radical

Definition 2. For $X \in \operatorname{Alg}\mathcal{N}$, define

$$i_N^+(X) := \inf_{M>N} \|(M-N)X(M-N)\|$$

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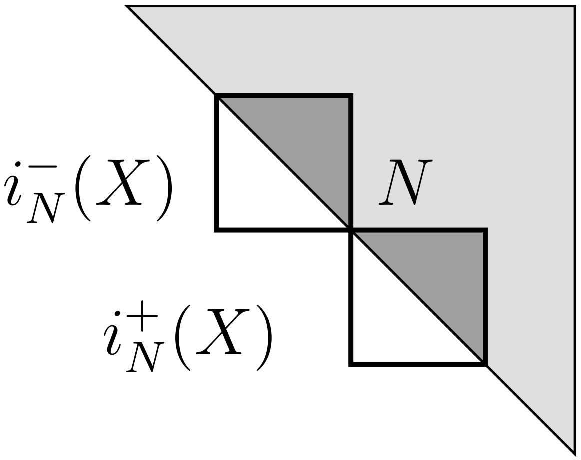

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Theorem 2 (Ringrose, '65). *The Jacobson Radical, $\mathcal{R}_{\mathcal{N}}$, of $\operatorname{Alg}\mathcal{N}$ is equal to*

$$\{X \in \operatorname{Alg}\mathcal{N} \ : \ i_N^+(X) = i_N^-(X) = 0 \quad \forall N \in \mathcal{N}\}$$

The Strong Radical

Let $\mathcal{N}$ be a *continuous* nest.

Theorem 3 (O., '94). *Used* i_N^+ *seminorms to classify the lattice of ideals generated by maximal two-sided ideals. Showed that the strong radical is*

$$\{X \in \operatorname{Alg}\mathcal{N} \ : \ i_N^+(X) = 0 \textit{ on a nowhere dense set}\}$$

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Remark 4. The strong radical for $\operatorname{Alg}\mathbb{Z}^+$ is unknown.

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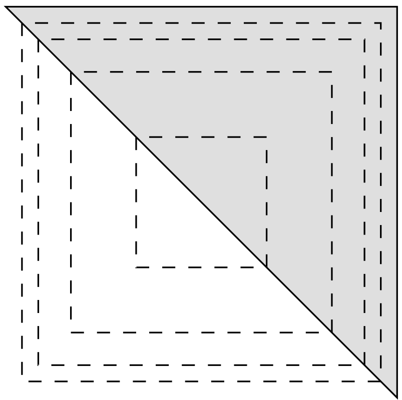
- Call $X \in \operatorname{Alg}\mathcal{N}$ *compact character* if $(M-N)X(M-N)$ is compact for all $0 < N < M < I$ in $\mathcal{N}$.



Compact Character

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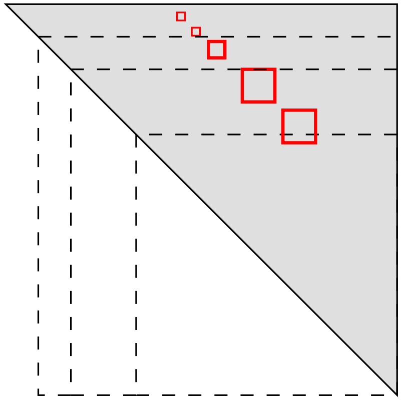
$$\mathcal{K}^{+} := \{X \in \operatorname{Alg}\mathcal{N} \ : \ N^{\perp} X N^{\perp} \in \mathcal{K} \quad \forall N > 0\}$$



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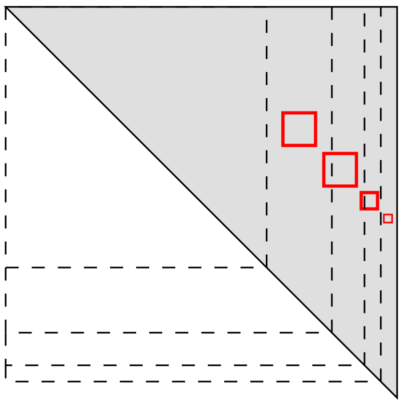

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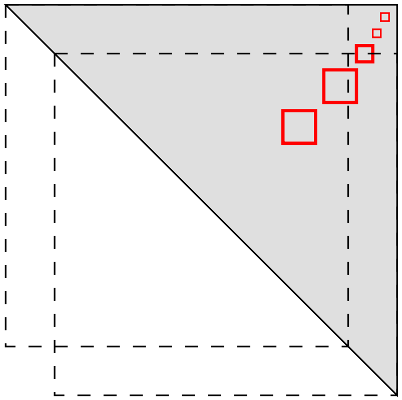
Example:

$$\mathcal{K}^- \cap \mathcal{K}^+$$



Stable Ideals

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From here on, all nests are continuous

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Examples:

- The trivial ideals 0 and $\operatorname{Alg}\mathcal{N}$
- The compact operators
- The set of operators of compact character
- The Jacobson radical
- The strong radical
- Many more...

Stable Ideals

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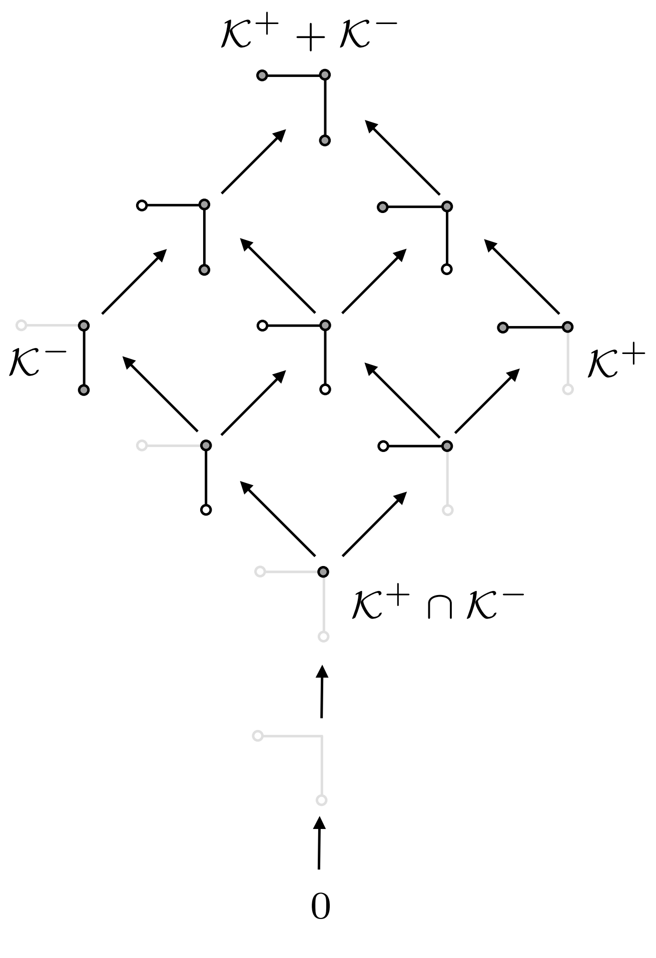
Non-Examples:

- Weakly closed ideals
- Larson's ideal, $\mathcal{R}_{\mathcal{N}}^{\infty}$

Stable Compact Char.



The lattice of 11 stable ideals of compact character

Automorphisms

Theorem 6 (Ringrose, '66). *Every isomorphism* $\mathrm{Alg}\,\mathcal{N}_1 \to \mathrm{Alg}\,\mathcal{N}_2$ *is of the form* Ad_S*, where* S *in an invertible operator.*

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Corollary 8. $\operatorname{Out}(\operatorname{Alg}\mathcal{N}) \longleftrightarrow \operatorname{Aut}([0,1])$

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- *It is one of the eleven stable ideals of compact character, or*
- *something horrid. . .*

Main Results

Main results:

- Simple, unified description of the stable ideals
- Classify the stable ideals
- Algebraic properties, quotient norms

Stable Nets

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Needn't even be countable!!

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Definition 11. Say that P_1 *refines* P_2 if whenever $E \in P_1$ there is an interval $F \in P_2$ such that $E \leq F$.

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Definition 11. A set, Ω, of families of interavls is a *net of intervals* if it is a directed set under this ordering. Ω is a *stable net* if

$$\theta(P) := \{\theta(E) \ : \ E \in P\} \in \Omega$$

for all $\theta \in \mathrm{Aut}([0,1])$.

Stable Nets & Ideals

Theorem 12 (O., preprint '05). *The (non-zero) set* $\mathfrak{I} \subseteq \operatorname{Alg}\mathcal{N}$ *is a stable ideal if and only if there is a stable net* Ω *such that* $\mathfrak{I}$ *is*

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But what does it *mean*?!

Examples

Example 13. Let Ω be just the one family, $P = \{0\}$. Then

$$\lim_{P \in \Omega} \sup_{E \in P} \|EXE\|_{\text{ess}} = 0$$

for all X. This gives the ideal $\mathcal{J} = \operatorname{Alg} \mathcal{N}$.

Examples

Example 13. Let Ω be just the one family, $P = \{I\}$. Then

$$\lim_{P\in\Omega} \sup_{E\in P} \|EXE\|_{\mathrm{ess}} = \|X\|_{\mathrm{ess}} = 0 \quad \Leftrightarrow \quad X \in \mathcal{K}$$

This gives the ideal $\mathcal{J} = \mathcal{K}$.

Examples

Example 13. Let Ω consist of all singletons $\{N\}$ with $N > 0$. Then

$$\lim_{P\in\Omega}\sup_{E\in P}\|EXE\|_{\mathrm{ess}} = \lim_{N\downarrow 0}\|NXN\|_{\mathrm{ess}} = i_0^+(X)$$

This gives the kernel of i_0^+.

Examples

Example 13. Let Ω consist of the single family $\{N \ : \ N < I\}$. Then

$$\lim_{P\in\Omega} \sup_{E\in P} \|EXE\|_{\text{ess}} = \sup_{N<I} \|NXN\|_{\text{ess}} = 0$$

$$\iff$$

$$X \in \mathcal{K}^{-}$$

Examples

Example 13. Let Ω consist of all finite partitions of $\mathcal{N}$. Then

$$\lim_{P\in\Omega}\sup_{E\in P}\|EXE\|_{\text{ess}} = \lim_{\{E_i\}}\sum_{i=1}^{n}\|E_iXE_i\| = 0$$

$$\iff$$

$$X \in \mathcal{R}_{\mathcal{N}}$$

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When do two stable nets give the same ideal?

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and so $\mathcal{J}_1 \supseteq \mathcal{J}_2$.

Classification Theorem

Theorem 14. *Let $\mathfrak{I}_1$ and $\mathfrak{I}_2$ be stable ideals associated with stable nets Ω_1 and Ω_2. Then $\mathfrak{I}_1 \supseteq \mathfrak{I}_2$ if and only if Ω_1 is* cofinal *in Ω_2.*

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Corollary 15. *$\mathfrak{I}_1 = \mathfrak{I}_2$ if and only if $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are mutually cofinal.*

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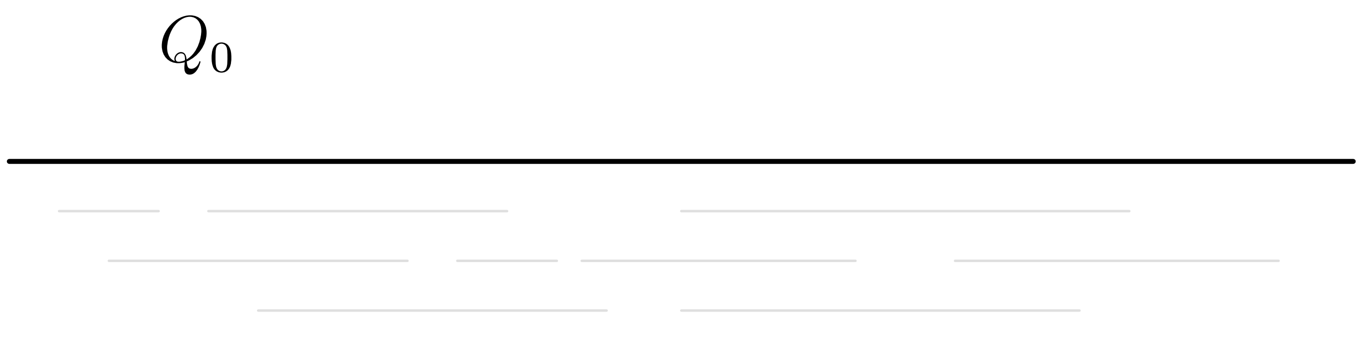

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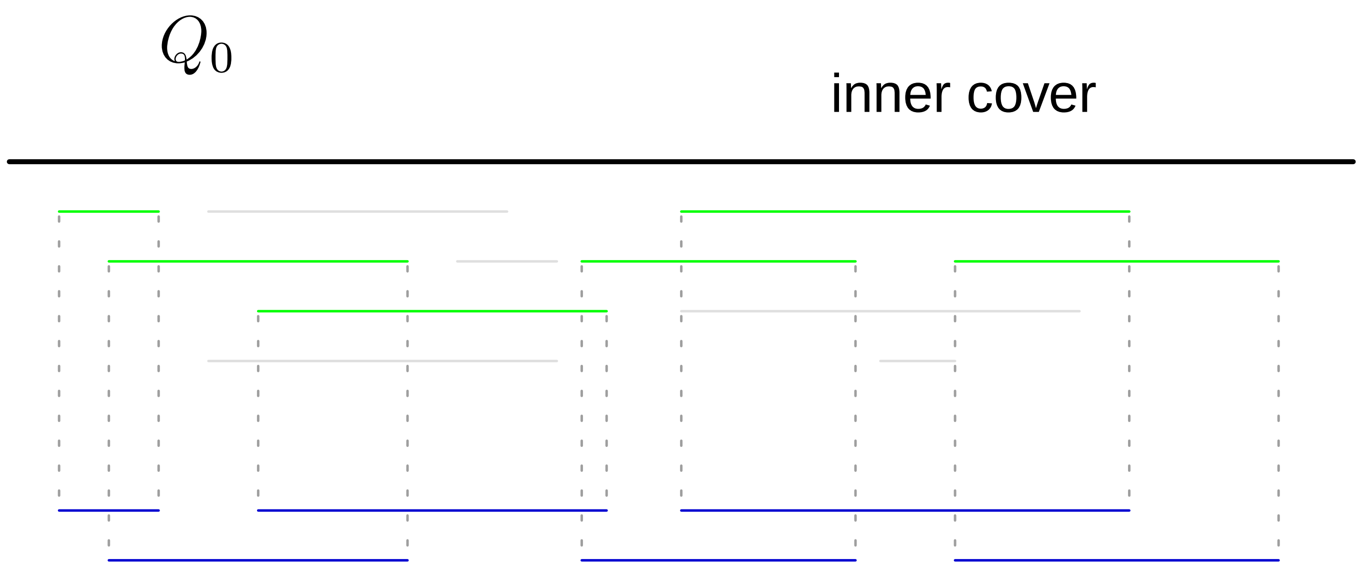

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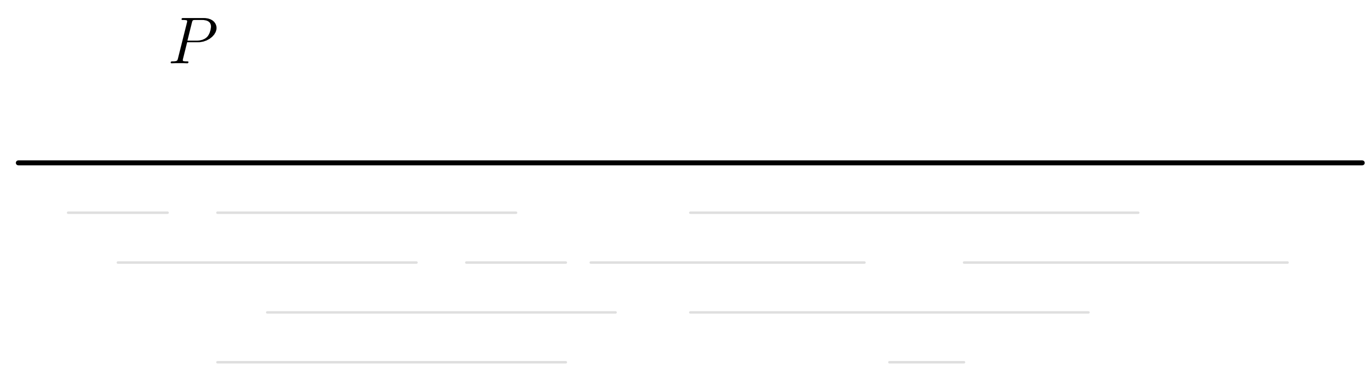

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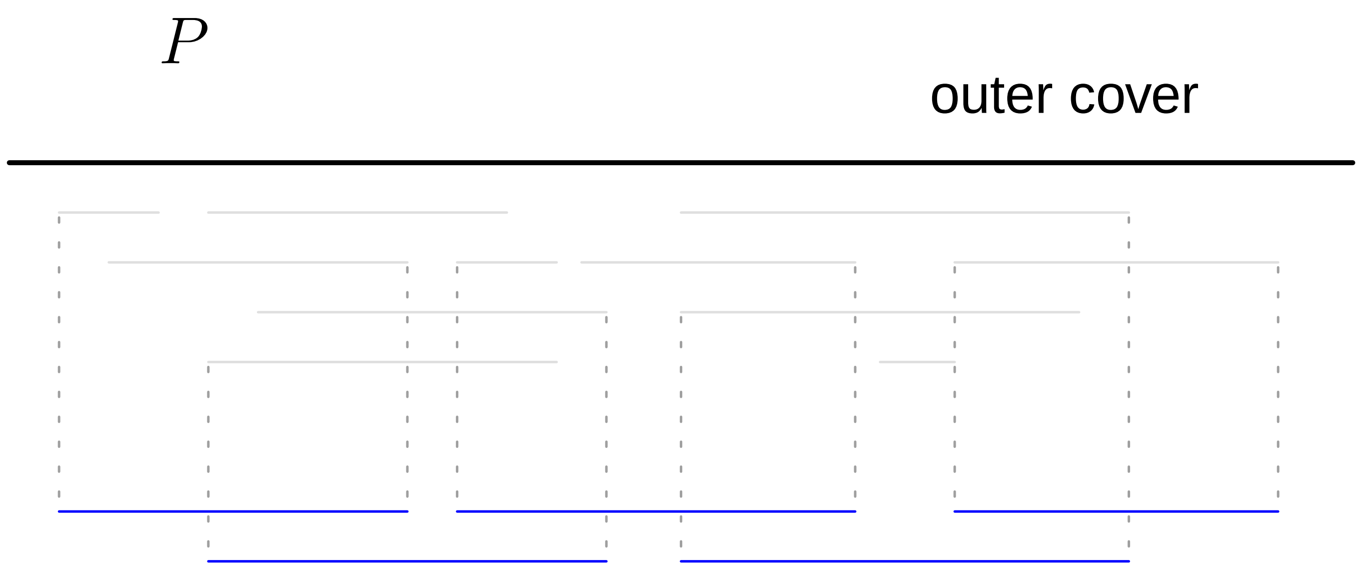

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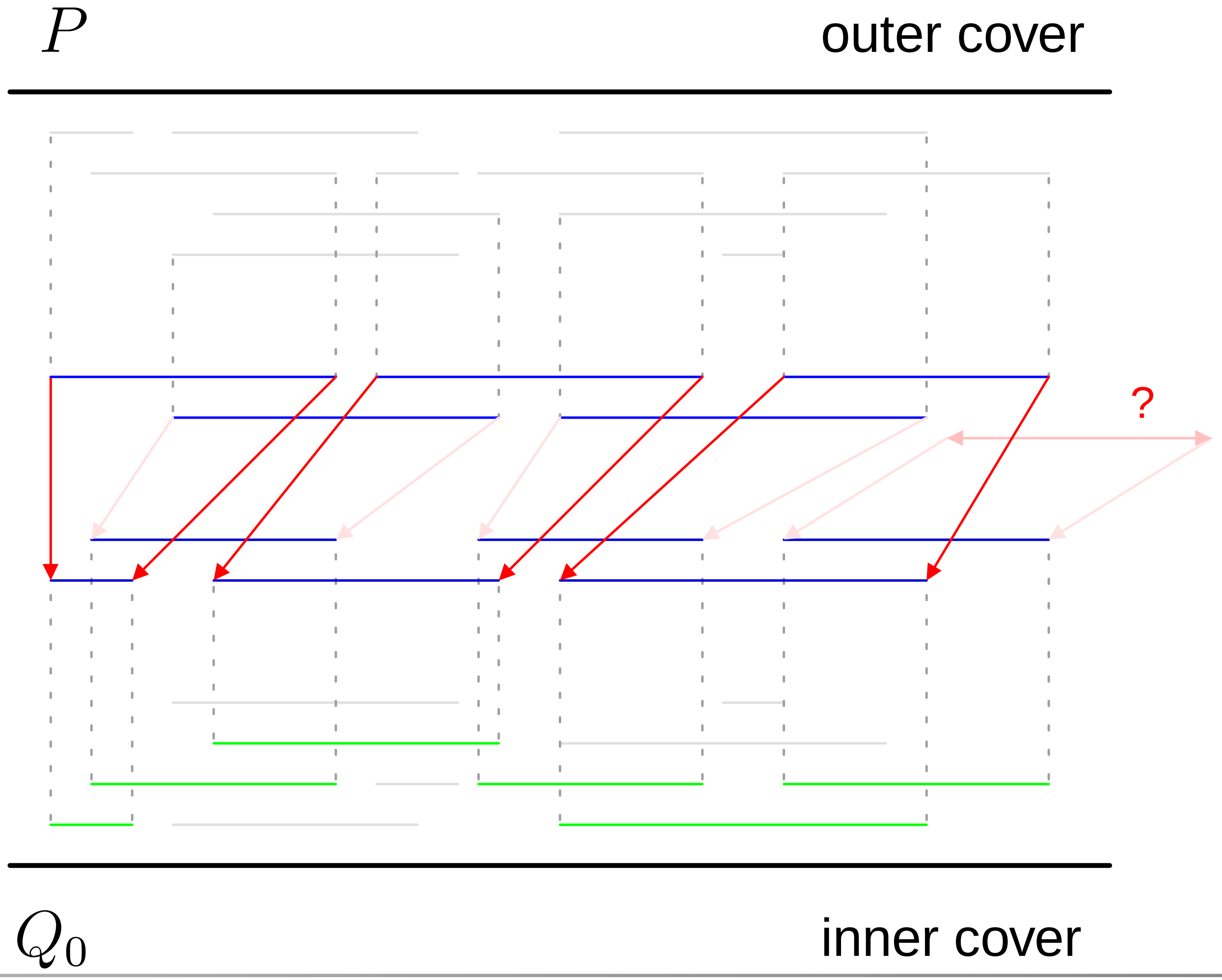
Match up the inner and outer covers. . .

Sketch of Proof



Quotient Norm

Theorem 16. *Let* $\mathfrak{J}$ *be given by* Ω *and* $X \in \operatorname{Alg}\mathcal{N}$*. Then*

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$$P_{T,a} := \{E \ : \ \|ETE < a\|_{\text{ess}}\} \quad T \in \mathfrak{J}, a > 0$$

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$$\Longrightarrow \lim_{P \in \Omega'} \sup_{E \in P} \|EXE\|_{\mathrm{ess}} = \lim_{P \in \Omega} \sup_{E \in P} \|EXE\|_{\mathrm{ess}}$$

Algebra of Ideals

Theorem 17. $\mathcal{J}_1, \mathcal{J}_2$ *stable ideals* $\implies \mathcal{J}_1 + \mathcal{J}_2$ *stable ideals.*

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How is net for $\mathcal{J}_1 + \mathcal{J}_2$ related to $\mathcal{J}_1, \mathcal{J}_2$?

Algebra of Ideals

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Let Ω_1, Ω_2 be stable nets. For $P_1 \in \Omega_1$ and $P_2 \in \Omega_2$ define

$$P_1 \cdot P_2 := \{E_1 E_2 \ : \ E_1 \in P_1, E_2 \in P_2\}$$

and then define

$$\Omega_1 \cdot \Omega_2 := \{P_1 \cdot P_2 \ : \ P_1 \in \Omega_1, P_2 \in O_2\}$$

Algebra of Ideals

Theorem 17. $\mathcal{J}_1, \mathcal{J}_2$ *stable ideals* $\implies$ $\mathcal{J}_1 + \mathcal{J}_2$ *stable ideals.*

Theorem 17. $\Omega := \Omega_1 \cdot \Omega_2$ *is a stable net, and*

$$\mathcal{J}_1 + \mathcal{J}_2 = \{X \in \operatorname{Alg}\mathcal{N} \ : \ \lim_{P \in \Omega} \sup_{E \in P} \|EXE\|_{\text{ess}} = 0\}$$